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Conifold singularities, resumming instantons and non-perturbative mirror symmetry

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Abstract: We determine the instanton corrected hypermultiplet moduli space in type IIB compactifications near a Calabi-Yau conifold point where the size of a two-cycle shrinks to zero. We show that D1-instantons resolve the conifold singularity caused by worldsheet instantons. Furthermore, by resumming the instanton series, we reproduce exactly the results obtained by Ooguri and Vafa on the type IIA side, where membrane instantons correct the hypermultiplet moduli space. Our calculations therefore establish that mirror symmetry holds non-perturbatively in the string coupling.

Keywords: String Duality, Supersymmetric Effective Theories.

Contents

1. Introduction

Recently, Calabi-Yau singularities have played a prominent role in "bottom-up" approaches connecting string theory to particle physics. The basic idea behind these constructions is to first locate a set of D-branes giving rise to the desired particle physics model at the singularity and later on perform the embedding into a compact Calabi-Yau threefold (CY_3). In this paper we study the simplest type of CY_3 singularities, the conifold, and determine the non-perturbative quantum corrections to the effective action for the (bulk) modulus which controls the size of the vanishing cycle. We expect that these type of corrections will become relevant when embedding the singularity into a full-fledged CY_3 compactification.

Geometrically, a conifold point is a point in the moduli space where the CY_3 becomes singular by developing a set of conical singularities (nodes) with base $S^2 \times S^3$. Locally these nodes can be resolved by either carrying out a deformation, by expanding the node into an S^3, or a small resolution expanding the node into an S^2. The process of shrinking a set of two-cycles S^2 to points and subsequently re-expanding the singularities into S^3's (or vice versa) is called a conifold transition and connects moduli spaces of CY_3 with different Hodge numbers [1, 2].

When approaching a conifold point by degenerating a complex structure ($S^3 \to 0$) and a Kähler structure ($S^2 \to 0$) in type IIB and type IIA string compactifications, respectively, the vector multiplet moduli space of the low energy effective action (LEEA) develops a logarithmic singularity. These singularities can be attributed to illegally integrating out D3 (D2) branes wrapping the vanishing cycles, which, at the conifold point, give rise to

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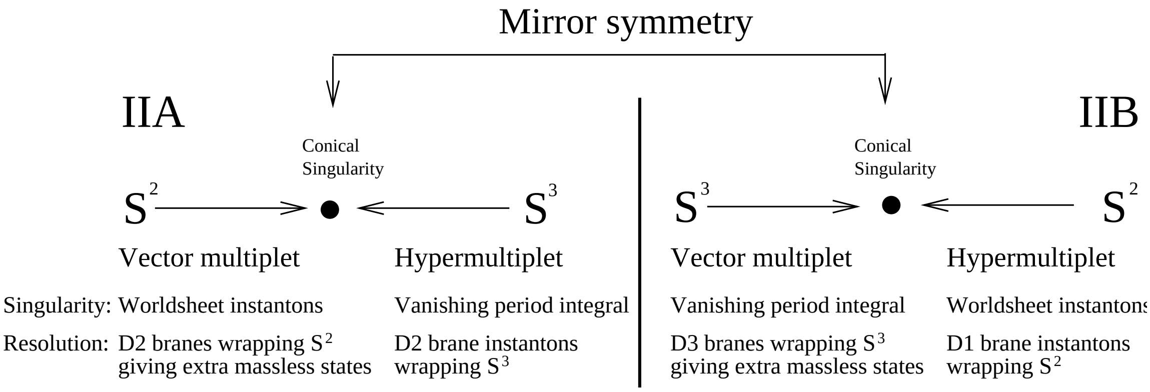

Figure 1: Illustration of the CY_3 moduli space close to a conifold point. For more information on conifold singularities we refer to [3, 6].

extra massless states [3]. On the other hand we can also approach the conifold point by degenerating a Kähler structure on the type IIB or a complex structure on the type IIA side. This leads to a logarithmic singularity in the hypermultiplet sector of the LEEA. In this case, however, the theory has no BPS states which could wrap the vanishing cycles and could have an interpretation in terms of four-dimensional particles. One expects, however, that non-perturbative string effects originating from instantons [4] become important in this regime since the real part of their instanton actions are proportional to the volume of the shrinking cycle so that they are no longer suppressed in the limit where the cycle shrinks to zero. See figure 1 for a schematic illustration. Indeed, as was shown in [5] in the context of IIA string theory, spacetime instanton effects survive in the effective field theory even in the rigid limit where gravity decouples, $M_{\rm Pl} \to \infty$.

More recently, new exact results were obtained in [7] for IIB strings, in which the contributions coming from worldsheet instantons, D1-instantons and D(-1)-instantons to the effective action were determined. Since these results are obtained at a generic point in the moduli space, we can study the behavior near the conifold point, where a two-cycle shrinks to zero size and gravity is decoupled. In this limit only worldsheet and D1-instanton corrections survive, and we obtain the resulting IIB hypermultiplet moduli space metric in the neighborhood of the conifold.

Our analysis allows us to perform a non-perturbative test of mirror symmetry, which states that the hypermultiplet moduli spaces in type IIA and type IIB on the mirror, after including all quantum corrections, must be the same. After resumming the instanton series on the IIB side, we determine the mirror map and show that the resulting hyperkähler geometry is exactly the one obtained in [5]. This provides a nice demonstration of open string mirror symmetry on the hypermultiplet moduli space.

2. IIA: summing up membrane instantons

In this section, we review the results of [5] who studied the geometry of the type IIA hypermultiplet (HM) moduli space near a conifold singularity associated with vanishing three-cycle $\mathcal{C}_3$,

$$\text{IIA HM conifold limit:} \quad z = \int_{\mathcal{C}_3} \Omega \to 0 \ . \tag{2.1}$$

To decouple gravity, we consider the combined limit

$$z \to 0\,, \quad \lambda \to 0\,, \quad \text{with } \frac{|z|}{\lambda} = \text{finite}\,, \tag{2.2}$$

where λ is the string coupling constant. In this limit, the moduli space becomes a four-dimensional hyperkähler space which at the classical level, develops a singularity at $z = 0$. The metric can be written as

$$\mathrm{d}s^2 = \lambda^2[V^{-1}(\mathrm{d}t - \vec{A} \cdot \mathrm{d}\vec{y})^2 + V|\mathrm{d}\vec{y}|^2] \ . \tag{2.3}$$

Here, $\vec{y} = (u, z/\lambda, \bar{z}/\lambda)$ with λ kept fixed and u and t are the RR scalars originating from the expansion of the RR 3-form with respect to the harmonic three-forms associated with the vanishing cycle $\mathcal{C}_3$ and its dual. The metric (2.3) is hyperkähler if

$$V^{-1}\Delta V = 0\,, \qquad \vec{\nabla} V = \vec{\nabla} \times \vec{A}\,, \tag{2.4}$$

where

$$\Delta = \partial_u^2 + 4\lambda^2 \partial_z \partial_{\bar{z}} \ . \tag{2.5}$$

Classically, at string tree-level and for large $|z|$, the metric is determined by

$$V = \frac{1}{4\pi} \ln\left(\frac{1}{z\bar{z}}\right), \qquad A_u = \frac{\mathrm{i}}{4\pi} \ln\left(\frac{z}{\bar{z}}\right)\,,\ A_z = 0\,,\ A_{\bar{z}} = 0\,, \tag{2.6}$$

and has a logarithmic singularity. Using T-duality, which exchanges vector multiplets and hypermultiplets, this singularity has a counterpart in the vector multiplet moduli space of the IIB theory compactified on the *same* Calabi-Yau threefold, where it corresponds to the appearance of massless black holes [3]. In fact, the hyperkähler metric (2.3) is related to the vector multiplet moduli space metric by the rigid c-map [8, 9]. We demonstrate this in appendix A. This will be important for us, since we use a similar mechanism for the mirror theory in the next section.

In [5] Ooguri and Vafa studied the resolution of the singularity based on D2-brane instanton contributions. Thereby they focused on the situation where the period with respect to $\mathcal{C}_3$ (A-cycle, say) vanishes while the dual period (from the B-cycle) remained finite. In this case membrane instantons wrapping the vanishing cycle generate exponential corrections to the hypermultiplet moduli space of the form $\exp(-|z|/\lambda)$ with θ-angle $\exp(2\pi\mathrm{i}u)$, breaking the shift symmetry in u to a discrete subgroup. Membrane instantons wrapping the dual B-cycle decouple in the rigid limit (2.2), so that the shift-symmetry in t is unbroken. The instanton corrected V was then found to be [5]

$$V = \frac{1}{4\pi} \ln\left(\frac{\mu^2}{z\bar{z}}\right) + \frac{1}{2\pi} \sum_{m\neq 0} K_0\left(2\pi \frac{|mz|}{\lambda}\right) \mathrm{e}^{2\pi\mathrm{i}mu}\,, \tag{2.7}$$

for some constant μ. This instanton sum contains the zero-th order modified Bessel function, accompanied by theta-angle-like terms set by the RR scalar u. The Bessel function

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can further be expanded for large argument, yielding exponentially suppressed terms of the form $\exp[-2\pi(|mz|/\lambda - imu)]$ together with an infinite power series in λ that describe the perturbative fluctuations around the instantons.

To exhibit the resolution of the singularity, one can perform a Poisson resummation,

$$V = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{(u-n)^2 + z\bar{z}/\lambda^2}} - \frac{1}{|n|} \right) + \text{const}\,, \tag{2.8}$$

which leads to a regular metric at $z = 0$.

In the case of N three-cycles shrinking to zero size, it was argued in [5, 6] that this leads to hyperkähler metrics with $\mathbb{C}^2/Z_N$ singularities. This metric is again of the form (2.3), with $V \to NV$. In the next sections, we will reproduce all these results from type IIB strings compactified on the mirror Calabi-Yau, in which N now counts the number of vanishing two-cycles. Thereby, we perform a non-perturbative test of mirror symmetry.

3. Conifold singularities in type IIA vector multiplets

To understand the origin of the conifold singularity in the IIB hypermultiplet moduli space, it is insightful to first study its counterpart on the type IIA vector multiplet side. The two sectors are related by T-duality and, at string tree-level, the IIB hypermultiplet moduli space is obtained from the IIA vector multiplet moduli space by the c-map [8, 10]. At a generic point in the moduli space, the special geometry is determined by a holomorphic prepotential $F(X)$ homogeneous of degree two, which receives perturbative α' corrections from the worldsheet conformal field theory and worldsheet instantons

$$F(X) = F_{\text{cl}}(X) + F_{\text{pt}}(X) + F_{\text{ws}}(X)\ . \tag{3.1}$$

Here

$$\begin{aligned} F_{\text{cl}}(X) &= \frac{1}{3!}\,\kappa_{abc}\,\frac{X^a X^b X^c}{X^1}\,, \qquad F_{\text{pt}}(X) = \mathrm{i}\,\frac{\zeta(3)}{2(2\pi)^3}\,\chi_E\,(X^1)^2\,, \\ F_{\text{ws}}(X) &= -\,\mathrm{i}\,\frac{1}{(2\pi)^3}\,(X^1)^2 \sum_{k_a} n_{k_a}\,\mathrm{Li}_3\left(\mathrm{e}^{2\pi\mathrm{i}k_a X^a/X^1}\right)\,, \end{aligned} \tag{3.2}$$

with κ_{abc}, χ_E and n_{k_a} the triple intersection numbers, Euler number and instanton numbers of the CY_3 [11] respectively (see, e.g., [12] for additional background).

Microscopically the scalar fields in the vector multiplet sector arise from expanding the Kähler form J and the ten-dimensional NS two-form $\hat{B}$ in terms of harmonic two-forms ω_a of the CY_3 [13],

$$\hat{B} = B_2 + b^a\,\omega_a\ , \quad J = t^a\,\omega_a\ , \quad a = 2,\ldots,h^{(1,1)}+1\,. \tag{3.3}$$

These fields are combined into complexified Kähler moduli

$$z^a = b^a + \mathrm{i}t^a\ = \frac{X^a}{X^1}\,. \tag{3.4}$$

It is useful to factor out X^1 and work with the holomorphic function $f(z)$ determined by

$$F(X) = (X^1)^2 f(z)\,, \tag{3.5}$$

which is also computed by the genus zero topological string amplitude.

We are now interested in the conifold limit of the prepotential given above. In the CY_3 geometry the conical singularity is obtained by shrinking the size of a holomorphic two-cycle $\mathcal{C}_\star$ to zero:

$$\text{geometrical conifold singularity:} \quad t^\star \to 0\,. \tag{3.6}$$

We note, however, that the condition (3.6) is not sufficient for causing a singularity in the vector multiplet moduli space. Here the singularity arises if the *complexified* Kähler modulus is taken to zero:

$$\text{moduli space conifold singularity:} \quad z^\star \to 0\,. \tag{3.7}$$

This implies that we can avoid hitting the singularity by giving a non-vanishing real part $b^\star$ to the complexified Kähler modulus. Thus in the moduli space the conifold singularity is a line of complex codimension one.[1]

We can now take the conifold limit (3.7) for the prepotentials (3.2). Henceforth we consider the case of a conical singularity where one particular complexified Kähler modulus $z^\star = k_a z^a$ (for one particular and fixed vector k_a) shrinks to zero,[2] while the others are frozen to constant values. By inspection one then finds that the second derivatives of f (determining the metric) arising from $f_{\rm cl}(z)$ and $f_{\rm pt}(z)$ are regular in this limit. Applying the expansion formula (B.10) to the worldsheet instanton contribution, one obtains (we denote $N = n_{k_a}$ for the fixed vector k_a)

$$f_{\rm ws}(z) = \frac{N}{4\pi \mathrm{i}}\, z^2 \, \ln(z) + \cdots\,, \tag{3.8}$$

where the dots give rise to regular contributions in the Lagrangian. Computing

$$\partial_z f_{\rm ws} = \frac{N}{2\pi \mathrm{i}}\, z \, \ln(z) + \cdots\,, \tag{3.9}$$

one finds that this is in precise agreement with the singular behavior found in the IIB vector multiplet sector when going to the conifold point by shrinking N Lagrangian three-cycles [3]. In this case, however, z is interpreted as a complex structure moduli arising from the periods of the holomorphic three-form of the CY_3.

The qualitative results of this section have already been discussed by Strominger [3], where it was argued that the conifold singularities in the type IIA vector multiplet sector originate from strong coupling effect involving worldsheet instantons.

[1] This is different from the five-dimensional case [14] where such singularities are of real codimension one, so that a generic trajectory moving on the moduli space will not be able to avoid the singularity.

[2] We will drop the "$\star$" in the following.

4. Conifold singularities in IIB hypermultiplets

In this section, we derive the conifold singularities that arise in the hypermultiplet moduli space of type IIB compactifications. As in section 2, this singularity can be obtained from the rigid c-map on the vector multiplet sector of the IIA theory. Here, we will rederive it in a different way, starting from a generic point in the (tree-level) hypermultiplet moduli space, and then taking the conifold limit in which gravity decouples. Our description of the hypermultiplet moduli space geometry uses the conformal tensor calculus combined with methods used in projective superspace. In this way, $4n$-dimensional quaternion-Kähler geometry can be reformulated in terms of $4(n+1)$-dimensional hyperkähler geometry. For some background material we refer to [15 − 20].

The tree-level hypermultiplet moduli space can be conveniently written down in projective superspace [21], in terms of a contour integral representation [22] of the superspace Lagrangian density

$$\mathcal{L}(v,\bar{v},x) = \mathrm{Im}\oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\, H(\eta^I(\zeta))\,, \tag{4.1}$$

in terms of $h_{1,2}+2$ $N=2$ tensor multipets

$$\eta^I(\zeta) = \frac{v^I}{\zeta} + x^I - \bar{v}^I\zeta\,, \tag{4.2}$$

consisting of $N=1$ real linear multiplets x^I and $N=1$ chiral multiplets v^I. The Lagrangian density satisfies

$$\mathcal{L}_{x^Ix^J} + \mathcal{L}_{v^I\bar{v}^J} = 0\,, \tag{4.3}$$

and expresses the fact that the dual hypermultiplets parameterize a hyperkähler manifold with $h_{1,2}+2$ commuting shift symmetries [22]. Tensor multiplets can be used because the hypermultiplet geometries we need to consider have enough commuting isometries.[3] The scalars of the tensor multiplets transform as a triplet under SU(2) R-symmetry

$$\vec{r}^{\,I} = \left[2\,v^I,\, 2\,\bar{v}^I,\, x^I\right]\,, \quad \vec{r}^{\,I}\cdot\vec{r}^{\,J} = 2v^I\bar{v}^J + 2v^J\bar{v}^I + x^Ix^J\,. \tag{4.4}$$

For a given prepotential $F(X)$ encoding the vector multiplet couplings, the dual tensor multiplet Lagrangian after the (local) c-map can be obtained by evaluating the following contour integral [18]

$$\mathcal{L}(v,\bar{v},x) = \mathrm{Im}\oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\,\frac{F(\eta^\Lambda)}{\eta^0}\,. \tag{4.5}$$

Here $\eta^I = \{\eta^0,\eta^\Lambda\}$ with η^0 being the conformal compensator. The contour integral is taken around one of the roots ζ_+ of $\zeta\eta^0$ and can be evaluated in a gauge invariant way [23][4]

$$\mathcal{L}(v,\bar{v},x) = -\frac{\mathrm{i}}{2r^0}\left(F(\eta_+^\Lambda) - \bar{F}(\eta_-^\Lambda)\right) = -\frac{\mathrm{i}}{2r^0}\left((\eta_+^1)^2 f(z) - (\eta_-^1)^2\bar{f}(\bar{z})\right)\,, \tag{4.6}$$

[3] This is correct in the absence of three-brane and five-brane instantons, which are not relevant for the purpose of this paper.

[4] In [24] a slightly more complicated formula for $\mathcal{L}$ has been given, taking into account the logarithmic singularity at $\zeta = 0$. The two expressions, however, only differ by terms linear in x^I and therefore lead to the same Lagrangian.

with

$$\eta_+^\Lambda = \eta^\Lambda(\zeta_+) = x^\Lambda - \frac{x^0}{2}\left(\frac{v^\Lambda}{v^0} + \frac{\bar{v}^\Lambda}{\bar{v}^0}\right) - \frac{r^0}{2}\left(\frac{v^\Lambda}{v^0} - \frac{\bar{v}^\Lambda}{\bar{v}^0}\right), \tag{4.7}$$

$\eta_- = (\eta_+)^*$ and $z^a = \eta_+^a/\eta_+^1$.

From the superspace Lagrangian density $\mathcal{L}$, one can compute a tensor potential [20]

$$\chi(v, \bar{v}, x) = -\mathcal{L}(v, \bar{v}, x) + x^I \mathcal{L}_{x^I}, \tag{4.8}$$

where $\mathcal{L}_{x^I}$ denotes the derivative with respect to x^I. Dualizing the tensors to scalars, this potential becomes the hyperkähler potential of the corresponding hyperkähler cone above the quaternion-Kähler manifold [16]. Therefore, this function determines the entire low-energy effective action. Using the homogeneity properties of $\mathcal{L}$, one can derive the identity

$$\frac{1}{2}\left(\chi_{x^I x^J} + \chi_{v^I \bar{v}^J}\right) = \mathcal{L}_{x^I x^J}\ . \tag{4.9}$$

The components $\mathcal{L}_{x^I x^J}$ then appear in the kinetic terms of the scalars x^I and v^I in the effective Lagrangian.

Close to the conifold locus $f(z)$ is given by (3.8). Substituting into (4.6) yields

$$\mathcal{L}^{\rm cf}(v, \bar{v}, x) = -\frac{N}{8\pi r^0}\left((\eta_+^1)^2\, z^2\, \ln(z) + (\eta_-^1)^2\, \bar{z}^2\, \ln(\bar{z})\right)\ . \tag{4.10}$$

Here, only one tensor multiplet ($v, \bar{v}, x$ with $x = k_a x^a$ etc.) captures the degrees of freedom, and all others are frozen to constants. As one can explicitly check, this function satisfies

$$\left(\partial_v \partial_{\bar{v}} + \partial_x^2\right) \mathcal{L}^{\rm cf}(v, \bar{v}, x) = 0\,. \tag{4.11}$$

This is precisely the constraint coming from rigid $N = 2$ supersymmetry and expresses the fact that the geometry is four-dimensional hyperkähler. This is consistent with the fact that in this limit, gravity is decoupled, and the target space of hypermultiplets becomes hyperkähler.[5]

The function $\mathcal{L}$ is not yet to be compared with the function V appearing in the hyperkähler metric (2.3). As shown in [22], the relation is (in the dilatation gauge $r^0 = 1$)

$$V = r^0 \mathcal{L}_{xx}\ . \tag{4.12}$$

Straightforward computation shows that, up to an additive constant,[6]

$$V = r^0 \mathcal{L}_{xx} = -\frac{N}{4\pi} \ln(z\bar{z})\,, \tag{4.13}$$

which precisely matches (2.6). This shows that at string tree-level, mirror symmetry works.

[5]The rigid limit in special Kähler geometry was studied in detail in [25]. It would be desirable to have a similar study for hypermultiplets.

[6]This additive constant contributes to the parameter μ in (2.7) and depends on the particular CY_3 under consideration.

5. IIB: resummation of D1-instantons

The starting point for including the D1-instantons is the modular invariant tensor potential [7]

$$\chi_{(1)} = -\frac{r^0 \tau_2^{1/2}}{(2\pi)^3} \sum_{k_a} n_{k_a} \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau+n|^3} \left(1 + 2\pi |m\tau+n|\, k_a t^a\right) \mathrm{e}^{-S_{m,n}} \,, \tag{5.1}$$

with instanton action

$$S_{m,n} = 2\pi k_a \big(|m\tau+n|\, t^a - \mathrm{i} m\, c^a - \mathrm{i} n\, b^a \big) \,. \tag{5.2}$$

The primed sum is taken over all integers $(m,n) \in \mathbb{Z}^2 \backslash (0,0)$. Here we used the notation and conventions as in [7], and adapted the normalization in such a way that it is consistent with the prepotential (3.2). The formula (5.1) contains all contributions coming from both worldsheet instantons (sum over n) and D1-instantons (sum over m), which are the only relevant configurations that survive in the conifold limit.[7] The instanton action contains the dilaton-axion complex

$$\tau = \tau_1 + \mathrm{i}\tau_2 = a + \mathrm{i}\mathrm{e}^{-\phi} \,, \tag{5.3}$$

and the string coupling constant is given by $\lambda = \mathrm{e}^{\phi}$. Furthermore, the c^a are RR scalars that generate the theta-angle like terms for the D1-instantons. The relation between the "microscopic" scalars τ, z^a, c^a and the scalars appearing in the tensor multiplets (4.2) is given by [26, 23, 7]

$$\tau = \frac{1}{(r^0)^2} \left(\vec{r}^{\,0} \cdot \vec{r}^{\,1} + \mathrm{i}\, |\vec{r}^{\,0} \times \vec{r}^{\,1}| \right) \,, \quad z^a = \frac{\eta_+^a}{\eta_+^1} \,, \quad c^a = \frac{(\vec{r}^{\,0} \times \vec{r}^{\,1}) \cdot (\vec{r}^{\,1} \times \vec{r}^{\,a})}{|\vec{r}^{\,0} \times \vec{r}^{\,1}|^2} \,. \tag{5.4}$$

Following the discussion in section 4 we compute the function $\mathcal{L}_{xx}$ arising from (5.1). The corresponding calculation can be simplified by noting that both $\chi_{(1)}$ and $\mathcal{L}_{xx}$ are invariant under local SU(2) R-symmetry. Thus we can adopt a particular SU(2) gauge, e.g., setting $x^0 = x^1 = 0, v^0 = \bar{v}^0$ and then taking the derivatives of (5.1) with respect to $x, v, \bar{v}$. Re-expressing the result in gauge invariant variables we find

$$\mathcal{L}_{xx}(x, v, \bar{v}) = \frac{N}{4\pi r^0} \sum_{m,n}{}' \frac{1}{|m\tau+n|} \, \mathrm{e}^{-2\pi\, (|m\tau+n|\, t - \mathrm{i} m c - \mathrm{i} n b)} \,. \tag{5.5}$$

To compare with the type IIA results obtained by Ooguri and Vafa we also have to take the conifold limit. On the type IIB side this corresponds to

$$t \to 0 \,, \quad b \to 0 \,, \quad \tau_2 \to \infty \,, \quad \text{with} \quad \tau_2 |b + \mathrm{i}t| = \text{finite} \,. \tag{5.6}$$

Taking this limit requires resumming the instanton corrections appearing in (5.5). For this purpose we split the double sum into the contributions coming from worldsheet instantons,

[7]Reference [7] also determined the contributions from D(-1) instantons. They yield exponential corrections of the type $\exp(-|m|\tau_2)$ and therefore vanish in the limit of vanishing string coupling. Similar arguments show that three-brane and five-brane instantons decouple in the conifold limit (5.6).

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$m = 0$, and the D1-instantons plus their bound states, $m \neq 0, n \in \mathbb{Z}$:

$$\mathcal{L}_{xx}(x, v, \bar{v}) = \frac{N}{4\pi r^0} \sum_{n\neq 0} \frac{1}{|n|} \mathrm{e}^{-2\pi(\,|n|\,t - inb)} + \frac{N}{4\pi r^0} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{|m\tau + n|}\, \mathrm{e}^{-2\pi(\,|m\tau+n|\,t - imc - inb)}\,. \tag{5.7}$$

The first term can be summed up easily

$$\begin{aligned} \frac{N}{4\pi r^0} \sum_{n\neq 0} \frac{1}{|n|} \exp\Big\{ -2\pi\left(|n|t - \mathrm{i}nb\right)\Big\} &= -\frac{N}{4\pi r^0} \ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}z}\right) + \mathrm{c.c.} \\ &\simeq -\frac{N}{4\pi r^0} \ln\left(z\bar{z}\right)\,, \end{aligned} \tag{5.8}$$

where we took the conifold limit $z = b + \mathrm{i}t \to 0$ in the second line. Observe that this expression precisely reproduces (4.13). In order to take the conifold limit in the second line of (5.7) we first carry out a Poisson resummation in n. Using the results of appendix B.2 we find

$$\begin{aligned} &\frac{1}{4\pi r^0} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{|m\tau + n|}\, \mathrm{e}^{-2\pi(\,|m\tau+n|\,t - imc - inb)} \\ &\quad = \frac{1}{2\pi r^0} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} K_0\left(2\pi|m\tau_2|\sqrt{t^2 + (b+n)^2}\right) \mathrm{e}^{2\pi i m(c - \tau_1(b+n))} \\ &\quad \simeq \frac{1}{2\pi r^0} \sum_{m\neq 0} K_0(2\pi\tau_2|mz|)\, \mathrm{e}^{2\pi \mathrm{i} m(c-\tau_1 b)}\,. \end{aligned} \tag{5.9}$$

Here we have taken the conifold limit (5.6) in the second step. Note that in this limit the sum over n localizes such that only the $n = 0$ part gives a non-zero contribution.

Combining (5.8) and (5.9), we then obtain the D1-instanton corrected $\mathcal{L}_{xx}$ in the conifold limit

$$N^{-1} V = r^0 \mathcal{L}_{xx} = \frac{1}{4\pi} \ln\left(\frac{1}{z\bar{z}}\right) + \frac{1}{2\pi} \sum_{m\neq 0} K_0(2\pi\tau_2|mz|)\, \mathrm{e}^{2\pi \mathrm{i} m(c-\tau_1 b)} \tag{5.10}$$

Comparing this to the instanton corrected function V on the type IIA side in (2.7), we find perfect agreement if we use the mirror map

$$\lambda = \tau_2^{-1}\,, \qquad z^{\mathrm{IIA}} = z^{\mathrm{IIB}}\,, \qquad u = \pm(c - \tau_1 b)\,. \tag{5.11}$$

The second relation states that, under mirror symmetry, the complex structure modulus z^{IIA} is equated to the complexified Kähler modulus z^{IIB} associated with the vanishing cycles, while the relation between u and $c - \tau_1 b$ is determined up to a sign only. Note that selecting the minus sign, eq. (5.11) is precisely the classical mirror map obtained in [27]. This shows that the classical mirror map does not receive quantum corrections once the conifold limit is taken.

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A. Conifold singularities from the rigid c-map

The leading term in the type IIB vector multiplet prepotential can be determined by monodromy arguments

$$f_{\rm ws}(z) = \frac{1}{4\pi \mathrm{i}}\, z^2 \,\ln(z) + \cdots\,, \tag{A.1}$$

where z is associated with the vanishing period of the holomorphic three-form of the CY_3. We now interpret (A.1) as the prepotential underlying a rigid special Kähler geometry. We can then use the rigid c-map [8, 9] to construct the dual hyperkähler metric. For a general (rigid) prepotential $F(X^I)$ the resulting metric reads (up to a trivial rescaling)

$$-\frac{1}{2}\mathrm{d}s^2 = \mathrm{i}\left(\mathrm{d}F_I\,\mathrm{d}\bar{X}^I - \mathrm{d}\bar{F}_I\,\mathrm{d}X^I\right) - N^{IJ}\left(\mathrm{d}B_I - F_{IK}\mathrm{d}A^K\right)\left(\mathrm{d}B_J - \bar{F}_{JL}\mathrm{d}A^L\right)\,, \tag{A.2}$$

where $F_I = \partial F/\partial X^I$ and N^{IJ} being the inverse of

$$N_{IJ} = -\mathrm{i}\left(F_{IJ} - \bar{F}_{IJ}\right)\,. \tag{A.3}$$

Evaluating (A.2) for the prepotential (A.1) then yields

$$\mathrm{d}s^2 = \frac{1}{2\pi}\ln(z\bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z} - \frac{\pi}{\ln(z\bar{z})}\left(\mathrm{d}B - \frac{1}{2\pi\mathrm{i}}\ln(z)\,\mathrm{d}A\right)\left(\mathrm{d}B + \frac{1}{2\pi\mathrm{i}}\ln(\bar{z})\,\mathrm{d}A\right)\,. \tag{A.4}$$

Changing coordinates

$$B = 2\lambda\, t\,,\;\; A = -2\lambda\, u\,, \tag{A.5}$$

one finds precisely the metric (2.3) obtained from the solution (2.6).

B. Polylogarithms and resummation techniques

In this appendix we collect various facts used in the main part of the paper by giving a brief introduction to polylogarithmic functions and Poisson resummation in sections B.1 and B.2, respectively.

B.1 Polylogology

We start by summarizing some properties of polylogarithmic functions, essentially following the appendix B of ref. [29]. For $0 < z < 1$, the k-th polylogarithm is defined via the series expansion

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} . \tag{B.1}$$

It can be analytically continued to a multivalued function on the complex plane. Polylogarithms with different values of k are related by

$$z\frac{d}{dz}\mathrm{Li}_k(z) = \mathrm{Li}_{k-1}(z) . \tag{B.2}$$

For $k = 1$ we have

$$\mathrm{Li}_1(z) = -\log(1-z) , \tag{B.3}$$

which we used to sum up (5.8). From the definition (B.1) we find

$$\mathrm{Li}_k(0) = 0 , \quad (k \in \mathbb{Z}) \quad \text{and} \quad \mathrm{Li}_k(1) = \zeta(k) , \quad \text{for} \quad k > 1 . \tag{B.4}$$

Polylogarithms at values z and $1/z$ are related through the connection formula [30],

$$\mathrm{Li}_k(z) + (-1)^k \,\mathrm{Li}_k(1/z) = -\frac{(2\pi \mathrm{i})^k}{k!} B_k \left(\frac{\log(z)}{2\pi \mathrm{i}}\right) , \tag{B.5}$$

where $B_k(\cdot)$ are the Bernoulli polynomials. For $\mathrm{Li}_3(z)$ this yields

$$\mathrm{Li}_3(z) - \mathrm{Li}_3(1/z) = -\frac{1}{6}\log^3(z) - \frac{\mathrm{i}\pi}{2}\log^2(z) + \frac{\pi^2}{3}\log(z) . \tag{B.6}$$

From the point of view of the main part of the paper, it is more natural to work with the variable x, $z = \mathrm{e}^x$. In this case (B.6) becomes

$$\mathrm{Li}_3(\mathrm{e}^x) = \mathrm{Li}_3(\mathrm{e}^{-x}) - \frac{1}{6}x^3 - \frac{\mathrm{i}\pi}{2}x^2 + \frac{\pi^2}{3}x . \tag{B.7}$$

The conifold point corresponds to $x = 0$. At this point the function $\mathrm{Li}_3(\mathrm{e}^{-x})$ has a logarithmic branch point

$$\mathrm{Li}_3(\mathrm{e}^{-x}) \simeq q(x)\log(x) + p(x) \quad \text{for } x \to 0 , \tag{B.8}$$

where $q(x)$ and $p(x)$ are power series

$$q(x) = \sum_{j=0}^{\infty} q_j x^j , \qquad p(x) = \sum_{j=0}^{\infty} p_j x^j . \tag{B.9}$$

Analytically continuing the ansatz (B.8) to $\mathrm{Li}_3(\mathrm{e}^x)$ using $\log(-x) = \log(x) + \mathrm{i}\pi$ and substituting into the connection formula (B.7) we obtain the following expansion for small x

$$\mathrm{Li}_3\left(\mathrm{e}^{-x}\right) = -\frac{1}{2}x^2\ln(x) + p(x) , \tag{B.10}$$

where $p(x) = \zeta(3) - \zeta(2)x + \frac{3}{4}x^2 + \frac{1}{12}x^3 + \mathcal{O}(x^4)$ is polynomial in x. With this identity it is then straightforward to determine the conifold limit of the prepotential (3.2).

B.2 Poisson resummation

Taking the conifold limit in the D1-brane instanton sector in section 5 requires a Poisson resummation in the worldsheet instanton number n. The technical details of this computation are collected in this appendix.

The basic ingredient for Poisson resummation is the following identity for the Dirac delta-distribution

$$\sum_{n\in\mathbb{Z}} \delta(y-na) = \frac{1}{a}\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi i n y/a}\,, \;\; a\in\mathbb{R}^+\,. \tag{B.11}$$

Multiplying with an arbitrary function $f(x+y)$ and integrating over $y\in\mathbb{R}$ gives the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} f(x+na) = \frac{1}{a}\sum_{n\in\mathbb{Z}} \tilde{f}(2\pi n/a)\,\mathrm{e}^{2\pi i n x/a}\,. \tag{B.12}$$

Here $f(x)$ and $\tilde{f}(k)$ are related by Fourier-transformation

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\,\mathrm{e}^{-ikx}\,, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,\tilde{f}(k)\,\mathrm{e}^{ikx}\,. \tag{B.13}$$

We now apply this resummation to the second term in (5.7). Comparing

$$\sum_{n\in\mathbb{Z}} \frac{1}{|m\tau+n|}\,\mathrm{e}^{-2\pi(|m\tau+n|\,t-inb)} = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{(m\tau_2)^2+(n+m\tau_1)^2}}\,\mathrm{e}^{-2\pi(\sqrt{(m\tau_2)^2+(n+m\tau_1)^2}\,t-inb)} \tag{B.14}$$

to the general formula (B.12) we identify

$$\tilde{f}(2\pi n) = \frac{2\pi}{(\alpha^2+(2\pi n+\gamma)^2)^{1/2}}\mathrm{e}^{-(\alpha^2+(2\pi n+\gamma)^2)^{1/2}t}\,, \tag{B.15}$$

together with $a=1, x=b, \alpha=2\pi m\tau_2$, and $\gamma=2\pi m\tau_1$. The (inverse) Fourier transform of (B.15) can be found using the following formula for Fourier cosine transformations [28]:

$$\int_0^{\infty} \mathrm{d}x\,(x^2+\alpha^2)^{-1/2}\,\mathrm{e}^{-\beta\,(x^2+\alpha^2)^{1/2}}\cos(xy) = K_0\left[\alpha(\beta^2+y^2)^{1/2}\right]\,. \tag{B.16}$$

Substituting the result back into (B.12) then establishes the identity

$$\sum_{n\in\mathbb{Z}} \frac{1}{|m\tau+n|}\,\mathrm{e}^{-2\pi(|m\tau+n|t-inb)} = 2\sum_{n\in\mathbb{Z}} K_0\left(2\pi|m\tau_2|(t^2+(b+n)^2)^{1/2}\right)\mathrm{e}^{-2\pi i m\tau_1(b+n)}\,. \tag{B.17}$$

This completes the derivation of the first step in (5.9).

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